Physics of Planetary Systems — Exercises
Suggested Solutions to Set 10

Problem 10.1 *(2 points)*

The reflected light of the planet in comparison to the star can be calculated taking the Bond albedo A, the illuminated cross section of the planet πR_p^2 and the luminosity of the star at this orbital distance d into accoount, to be:

$$\frac{L_\mathrm{ref}}{L_*} = A\pi R_\mathrm{p}^2 \frac{L_*}{4\pi d^2}\frac{1}{L_*} = \frac{AR_\mathrm{p}^2}{4d^2},$$

where distance (or semi-major axis) can be related to orbital period via Kepler's third law:

$$P^2 = 4\pi^2 \frac{a^3}{G\mathscr{M}_*} \implies d = a = \left[G\mathscr{M}_* \left(\frac{P}{2\pi}\right)^2\right]^{1/3} \implies d[\mathrm{au}] = \mathscr{M}_*[\mathscr{M}_\odot]^{1/3} P[\mathrm{yr}]^{2/3}.$$

With $P = 0.79$ d $= 2.16 \times 10^{-3}$ yr and $\mathscr{M}_* = 0.95\mathscr{M}_\odot$, we find

$$d \approx 0.016 \text{ au} = 2.4 \times 10^6 \text{ km},$$

and with $R_\mathrm{p} = 1.3\,R_\mathrm{Jup}$, $R_\mathrm{Jup} \approx 70\,000$ km, and $A = 0.1$, the resulting luminosity ratio is

$$\frac{L_\mathrm{ref}}{L_*} = 3.45 \times 10^{-5}.$$

The ratios of thermal (blackbody) emission are given by:

$$\frac{L_{\mathrm{rad},\lambda}}{L_{*,\lambda}} = \frac{e^{hc/k\lambda T_*} - 1}{e^{hc/k\lambda T_\mathrm{p}} - 1}\frac{R_\mathrm{p}^2}{R_*^2},$$

where the planatary equilibrium temperature is

$$T_\mathrm{p} = T_* \sqrt{\frac{R_*}{d}} \left(\frac{1-A}{4}\right)^{1/4} = 1930 \text{ K},$$

i. e. very high. We find a radiated luminosity ratio of

$$\frac{L_{\mathrm{rad},\lambda}}{L_{*,\lambda}} = 1.3 \times 10^{-3},$$

which is approximately 38 times more than the reflected light. So if you want to detect the light of the planet, look in the infrared!

Alternatively you can use directly Planck's law to calculate the radiated and reflected power at wavelength λ:

$$L_{\mathrm{rad},\lambda} = \frac{2\pi hc^2/\lambda^5}{e^{hc/k\lambda T_\mathrm{p}} - 1} \cdot 4\pi R_\mathrm{p}^2, \quad L_{\mathrm{ref},\lambda} = A\frac{L_{*,\lambda}}{4\pi d^2} \cdot \pi R_\mathrm{p}^2, \quad \text{and} \quad L_{*,\lambda} = \frac{2\pi hc^2/\lambda^5}{e^{hc/k\lambda T_*} - 1} \cdot 4\pi R_*^2,$$

with $L_{*,\lambda}$ being the radiated power by the star. Comparing these two formulas directly, without the need to calculate the individual values,

$$\frac{L_{\mathrm{rad},\lambda}}{L_{\mathrm{ref},\lambda}} = \frac{e^{hc/k\lambda T_*} - 1}{e^{hc/k\lambda T_\mathrm{p}} - 1} \cdot \frac{4d^2}{AR_*^2}$$

and using the above calculated values for the equilibrium temperature $T_\mathrm{p} = 1930$ K and the orbital distance $d \approx 0.016$ au, as well as albedo $A = 0.1$ and wavelength $\lambda = 2$ µm, we find again a ratio of approximately 38 in favour of the radiated light.

Problem 10.2 *(2 points)*

Again we can use the formula for the equilibrium temperature of a planet, however, this time solving for the planet distance, hence in the following form:

$$d = \left(\frac{T_*}{T_p}\right)^2 R_* \left(\frac{1-A}{4}\right)^{1/2} \qquad \text{or} \qquad d[\text{au}] = \left(\frac{279\ \text{K}}{T_\text{p}}\frac{T_*}{T_\odot}\right)^2 \frac{R_*}{R_\odot}\sqrt{1-A},$$

where $T_\text{p} = 323$ K, $T_\text{p} = 3170$ K, $T_\odot = 5780$ K, $R_* = 0.27R_\odot$, and $A \approx 0.3$, resulting in

$$d \approx 0.05\ \text{au}.$$

In a next step the radial velocity (RV) amplitude is of interest. First one has to calculate the orbital Period from Kepler's third law (with $a = d$):

$$P = 7.85\ \text{d}.$$

Now we can use the formula for the RV amplitude for v_obs (see lecture notes):

$$v_\text{obs} = 28.4\ \text{m/s} \times \frac{\mathscr{M}_\text{p}[\mathscr{M}_\text{Jup}]\sin i}{P[\text{yr}]^{1/3}\mathscr{M}_*[\mathscr{M}_\odot]^{2/3}} < 0.9\ \text{m/s},$$

where $\mathscr{M}_\text{p} = \mathscr{M}_\oplus = 3\times10^{-3}\mathscr{M}_\text{Jup}$ and $\mathscr{M}_* = 0.27\mathscr{M}_\odot$. Modern spectrographs can get an RV precision of 0.5...1 m/s, so yes, this planet is detectable by modern spectrographs (unless $i \approx 0$), which is why we look for (low-mass) planets around M dwarfs.

Problem 10.3 *(1 point)*

As discussed in the lecture, the migration rate for type-II migration in the low-mass and high-mass regimes is given by

$$\dot{a}_\text{p} \approx \begin{cases} v_r & \text{(low mass)} \\ v_r\sqrt{\frac{4\pi\Sigma a_\text{p}^2}{\mathscr{M}_\text{p}}} & \text{(high mass)}, \end{cases} \tag{1}$$

where Σ is the surface mass density of the surrounding gas disk, v_r its radial drift speed, and a_p and $\mathscr{M}_\text{p}$ the planetary semi-major axis and mass, respectively. (The corresponding timescale could be defined as $t_\text{II} \equiv a_\text{p}/\dot{a}_\text{p}$.) The two regimes are joined where the square-root term becomes unity, i. e.

$$\mathscr{M}_\text{p} = 4\pi\Sigma a_\text{p}^2. \tag{2}$$

Below that critical mass, the planet is less massive than the surrounding disk. It will be dragged along with the viscous gas that slowly drifts toward the star. Above the critical mass, the planet is more massive than the disk and will not be affected as strongly, resulting in a slower migration.

Assuming $a_\text{p} = 1\ \text{au} = 1.5\times10^{13}$ cm and $\Sigma = 1000\ \text{g cm}^{-2}$, we find

$$\mathscr{M}_\text{p} \approx 2.8\times10^{30}\ \text{g} \approx 1.5\ \mathscr{M}_\text{Jup}, \tag{3}$$

which is quite a lot.

Bonus problem 10.4 *(3 extra points)*

If the material from both sides of the gap can reach the gap center within one orbital period, the gap is closed. Thus, we have to equate the time the planet needs to reach the same position relative to the gas again with the time needed for the gas to traverse a distance equal to the planet's Hill radius r_H:

$$\frac{2\pi r}{v_\text{rel}} = \frac{r_\text{H}}{v_\text{fill}}, \tag{4}$$

where $v_{\rm fill}$ is the speed at which the gas can refill the gap and $v_{\rm rel}$ is the difference between the tangential velocities of gas and embryo:

$$v_{\rm rel} = v_{\rm K} - v_{\rm gas} = v_{\rm K}\,(1 - \sqrt{1-2\eta}) \approx \eta v_{\rm K},$$

with $\eta \equiv c_{\rm s}^2/v_{\rm K}^2$. Using

$$r_{\rm H} = r\left(\frac{\mathscr{M}_{\rm p}}{3\mathscr{M}_*}\right)^{1/3}, \tag{5}$$

we arive at

$$\frac{2\pi r}{v_{\rm rel}} = \frac{r}{v_{\rm fill}}\left(\frac{\mathscr{M}_{\rm p}}{3\mathscr{M}_*}\right)^{1/3},$$

and solving for $\mathscr{M}_{\rm p}$ leads to

$$\mathscr{M}_{\rm p} = 3\mathscr{M}_*\left(\frac{2\pi v_{\rm fill}}{v_{\rm rel}}\right)^3.$$

Now, we can assume a filling velocity that equals the radial drift velocity:

$$v_{\rm fill} = \frac{3\nu}{2r} = \frac{3\alpha c_{\rm s}^2}{2r\Omega_{\rm K}} = \frac{3\alpha c_{\rm s}^2}{2v_{\rm K}}, \tag{6}$$

from which we obtain

$$\mathscr{M}_{\rm p} = 3\mathscr{M}_*\left(\frac{3\pi\alpha c_{\rm s}^2}{\eta v_{\rm K}^2}\right)^3 = 3\mathscr{M}_*\,(3\pi\alpha)^3 \approx 2500\alpha^3\mathscr{M}_*.$$

Assuming $\alpha = 0.001\ldots0.01$ and $\mathscr{M}_* = M_\odot$ leads to

$$\begin{aligned}\mathscr{M}_{\rm p} &= 2.5\times10^{-6}\mathscr{M}_*\ldots2.5\times10^{-3}\mathscr{M}_*\\ &\approx 1\,M_\oplus\ldots2\,M_{\rm Jup}.\end{aligned}$$