Physics of Planetary Systems — Exercises
Suggested Solutions to Set 7

Problem 7.1 *(2 points)*

The scattering of a laser beam tuned to the sodium D lines of the mesopheric sodium layer can be used for sensing of the wavefront disturbances introduced by the atmosphere. Subsequently, adaptive optics (AO) corrects the wavefront with deformable mirrors to remove the atomspheric seeing.

Assume that the laser illuminates a sodium-rich layer with a thickness $\delta h_{\rm Na} = 11.5\,\rm km$ at an altitude of $h_{\rm Na} = 90\,\rm km$ (Fig. 1). If d denotes the distance between the telescope and the laser emitter, the pointing offset for the laser is $\alpha = d/h_{\rm Na}$. From a simple geometrical consideration, the angular diameter of the artificial guide star is

$$\beta_{\rm laser} = \frac{\alpha\ \delta h_{\rm Na}}{h_{\rm Na}} = \frac{d\ \delta h_{\rm Na}}{h_{\rm Na}^2} = 206265''\ \frac{d\ \delta h_{\rm Na}}{h_{\rm Na}^2}.$$

If the emitter is mounted on the rim of a telescope with a diameter of 4 m, i.e. $d = 2\,\rm m$, the result is $\beta_{\rm laser} = 0.59''$. While this width appears to be bad for observing the artificial star in the AO system, it is actually not a big problem because the measured position (in terms of the photocenter) is not as sensitive to the width on sky.

Extra info: the aperture of the laser optics will also influence the size of the "star". As always, the diffraction limit of the laser projector is given by $\beta_{0,\rm laser} \approx 1.2\lambda_{\rm laser}/D_{\rm laser}$, which is then the apparent size of the artificial guide star as seen from the ground. Here, $\lambda_{\rm laser}$ is the wavelength at which the laser operates and $D_{\rm laser}$ the diameter of its aperture (0.3 m for the VLT's laser facility). The minimum apparent size is given by

$$\beta_{0,\rm laser} \approx 1.2\frac{\lambda_{\rm laser}}{D_{\rm laser}} \approx 1.2\frac{0.5\ \mu\rm m}{0.3\ \rm m} = 0.4'', \tag{1}$$

which is already comparable with the above $\beta_{\rm laser}$. So, in total, the artificial star is an elongated spot that covers an area $0.4'' \times (0.4'' + 0.6'') = 0.4'' \times 1.0''$ wide.

Problem 7.2 *(1 point)*

For the direct-imaging technique, which is complementary to most of the other techniques, as it works best for large planet separations, we have the best chances to find planets around young, nearby stars in order to keep the luminosity quotient between star and planet as well as the projected separation of close planets at a favorable level. The companion of β Pictoris at a distance of 19.3 pc within a young association is a very good example for these advantages.

First we have to determine at which projected angle β Pictoris b could be found by using the Rayleigh criterion (characterising the diffraction limit)

$$\alpha_{\rm R}(\text{a.k.a. } \beta_0) = 1.22\ \frac{\lambda}{D}$$

at the given wavelength 2.2 µm and the size of the VLT of 8.2 m as well as the fact that the companion was found at 3 times this diffraction limit

$$\alpha_{\rm VLT} = 3\alpha_{\rm R} = 3.66\ \frac{\lambda}{D} = 3.66\ \frac{2.2\ \mu\rm m}{8.2\ \rm m} \approx 9.8 \times 10^{-7}\rm rad \approx 0.2''.$$

Taking into account that this corresponds to the projected separation between star and planet at the distance $d = 19.3$ pc of the stellar primary, we can calculate the projected separation a of the planet in au:

$$a = d\ \tan\alpha_{\rm VLT} \approx 4\,\rm au.$$

We can repeat that for the given parameters of the Extremely Large Telescope (ELT) and find

$$\alpha_{\rm ELT} = 3\alpha_{\rm R} = 3.66\ \frac{\lambda}{D} = 3.66\ \frac{2.2\ \mu\rm m}{39\ \rm m} \approx 1.9 \times 10^{-7}\rm rad \approx 0.04''$$

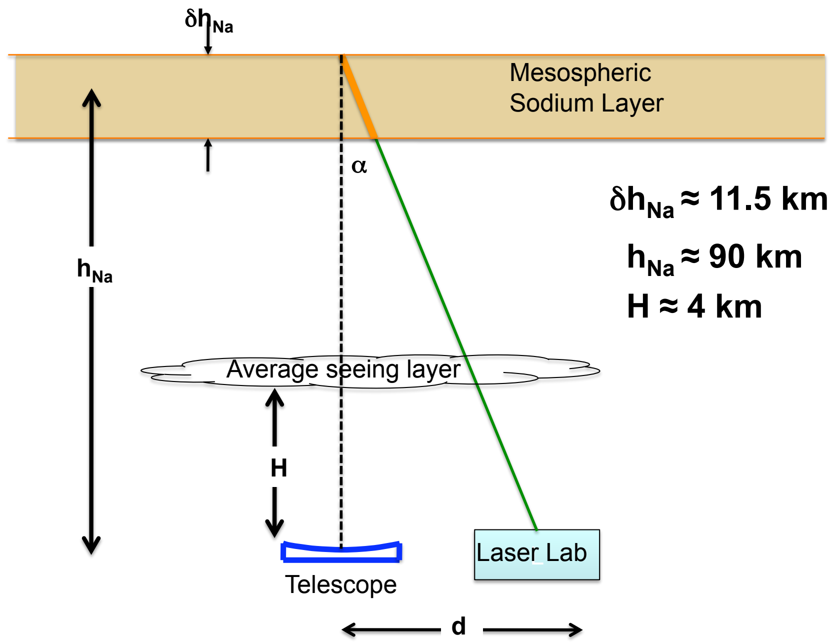

Figure 1: Geometry of the problem: telescope and laser emitter on the ground, artificial laser guide star in the mesosphere. The average seeing layer is in between, relatively close to the ground.

and, hence

$$a = d\ \tan\alpha_{\mathrm{ELT}} \approx 0.8\,\mathrm{au}.$$

This value is of course only reachable if future adaptive optics (AO) systems reach the same performance as for the VLT or better.

Extra info: The reduction of the possible imageable separation might not seem to change by a huge amount from 4 to 0.8 au or 20 % of the value of the VLT, corresponding to the increase in mirror size. However, in addition to the fact that high-mass planets will be imageable at distances comparable to the Astronomical Unit, the orbital period of such planets decreases (around an object of solar mass for example) from > 7.7 years to > 242 days, corresponding to a reduction to only 8.6 % of the original 7.7 years. Hence, the ELT will not only allow to find and follow planets along their shorter orbits, but will help to combine several planet search techniques for the same closer in planets, as e. g. imaging, radial velocity, astrometry and possibly also the transit technique.
Lecavelier Des Etangs et al. (2009) investigated the possibility that β Pictoris b might be a transiting planet, as in November 1981 strong and rapid photometric variations were observed for β Pictoris, that were attributed to the transit of a giant comet or a planet orbiting at several au.

Problem 7.3 *(2 points)*

To be able to integrate the growth, it is easier to first convert from masses to radii. The relation

$$m = \frac{4\pi\rho s^3}{3} \tag{2}$$

corresponds to

$$\dot{m} = 4\pi\rho s^2 \dot{s}. \tag{3}$$

This can be combined with the given growth rate,

$$\dot{m} = -\text{const} \times s^2 \rho_{\text{gas}} \dot{z}, \tag{4}$$

to obtain

$$4\pi\rho s^2 \dot{s} = -\text{const} \times s^2 \rho_{\text{gas}} \dot{z}, \tag{5}$$

and hence,

$$4\pi\rho \mathrm{d}s = -\text{const} \times \rho_{\text{gas}} \mathrm{d}z. \tag{6}$$

Note that the constant is negative because the altitude z decreases. Both sides of the equation can be integrated separately, resulting in

$$4\pi\rho\left(s_{\text{final}} - s_0\right) = -\text{const} \times \int_{z_0}^{0} \rho_{\text{gas}} \mathrm{d}z \approx \text{const} \times \frac{\Sigma_{\text{gas}}}{2}, \tag{7}$$

because the column density (or surface mass density) is related to the volume mass density via

$$\Sigma_{\text{gas}} = \int_{-\infty}^{\infty} \rho_{\text{gas}} \mathrm{d}z, \tag{8}$$

and $z_0 \approx \infty$. Solving for the final grain radius, we obtain

$$s_{\text{final}} = s_0 + \text{const} \times \frac{\Sigma_{\text{gas}}}{8\pi\rho}. \tag{9}$$

As long as the initial radius is small ($0 \approx s_0 \ll s_{\text{final}}$), the final radius is independent from it:

$$s_{\text{final}} \approx \text{const} \times \frac{\Sigma_{\text{gas}}}{8\pi\rho}. \tag{10}$$

Bonus problem 7.4 *(2 extra points)*

Deriving an actual value for s_{final} requires estimates for Σ, ρ, and in particular, "const". For the gas column density and the grain bulk density we can assume typical values (at ~ 1 au):

$$\Sigma_{\text{gas}} \sim 1000 \text{ g/cm}^2, \quad \rho \sim 1 \text{ g/cm}^3. \tag{11}$$

The constant can be obtained from a comparison with the growth equation given in the lecture:

$$\dot{m} = \sigma \rho_{\text{dust}} v_{\text{sett}}, \tag{12}$$

where $v_{\text{sett}} = -\dot{z}$, $\sigma = \pi s^2$, and $\rho_{\text{dust}} \sim 0.01\rho_{\text{gas}}$. Hence we find

$$\dot{m} \sim -0.01\pi s^2 \rho_{\text{gas}} \dot{z}, \tag{13}$$

and comparison with equation (4) shows that

$$\text{const} \sim 0.01\pi. \tag{14}$$

With these numbers, the final radius is then given by

$$s_{\text{final}} \sim 0.01\pi \times \frac{1000 \text{ g/cm}^2}{8\pi \times 1 \text{ g/cm}^3} \sim \frac{10}{8} \text{ cm} \sim 1 \text{ cm}. \tag{15}$$

Problem 7.5 *(1 point)*

On Earth's surface, the vertical free-fall acceleration is given by

$$\ddot{z} = -g \quad \overset{\int \ldots \mathrm{d}t}{\Longrightarrow} \quad \dot{z}(t) - \dot{z}(0) = -gt \quad \overset{\int \ldots \mathrm{d}t}{\Longrightarrow} \quad z(t) - z(0) - t\dot{z}(0) = -\frac{1}{2}gt^2, \tag{16}$$

where z is height, $\dot{z}$ vertical speed, and $g \approx 9.8\ \mathrm{m/s^2}$ the free-fall constant (which we assume to be constant). Hence, we find

$$z(0) = z(t) - t\dot{z}(0) + \frac{1}{2g}\left[\dot{z}(0) - \dot{z}(t)\right]^2, \tag{17}$$

which can be simplified to

$$z(0) = \frac{1}{2g}\left[\dot{z}(t)\right]^2 \tag{18}$$

if we assume start at rest ($\dot{z}(0) = 0$) and finish on the ground ($z(t) = 0$). For $\dot{z}(t) = -1$ cm/s, the required initial altitude is then

$$z(0) \approx \frac{1}{20\ \mathrm{m/s^2}}\left[0.01\ \mathrm{m/s}\right]^2 = 5\ \mu\mathrm{m}, \tag{19}$$

which is surprisingly small!