Physics of Planetary Systems — Exercises
Suggested Solutions to Set 8

Problem 8.1 *(2 points)*

We first need to compute the Einstein Radius, θ_E,

$$\begin{aligned}\theta_\mathrm{E} &= \sqrt{\frac{4G\mathscr{M}}{c^2}\frac{D_\mathrm{LS}}{D_\mathrm{L}D_\mathrm{S}}} = 1.38\times10^{-8}\ \mathrm{rad}\times\sqrt{\mathscr{M}[\mathscr{M}_\odot]\frac{D_\mathrm{LS}[\mathrm{kpc}]}{D_\mathrm{L}[\mathrm{kpc}]\ D_\mathrm{S}[\mathrm{kpc}]}} \\ &= 2.85\ \mathrm{mas}\times\sqrt{\mathscr{M}[\mathscr{M}_\odot]\frac{D_\mathrm{LS}[\mathrm{kpc}]}{D_\mathrm{L}[\mathrm{kpc}]\ D_\mathrm{S}[\mathrm{kpc}]}}\end{aligned} \tag{1}$$

We then need to calculate the magnification from:

$$\mu = \frac{u^2+2}{u\sqrt{u^2+4}},$$

where u is defined as $u \equiv \beta/\theta_\mathrm{E}$, and β is the impact parameter in radians. Asymptotically, μ can be approximated as

$$\mu \to \begin{cases} 1+\frac{2}{u^4} \text{ for } u \gg 1, \\ \frac{1}{u} \text{ for } u \ll 1. \end{cases} \tag{2}$$

The duration of the event is given by

$$t = \frac{R_\mathrm{E}}{v} = \frac{\theta_\mathrm{E}D_\mathrm{L}}{v},$$

with $R_\mathrm{E} = \theta_\mathrm{E}D_\mathrm{L}$ being the the projected Einstein Radius. The involved distances are $D_\mathrm{L} = 2$ kpc, $D_\mathrm{S} = 10$ kpc, $D_\mathrm{LS} = 8$ kpc.

a) Assuming $\mathscr{M} = 1\ \mathscr{M}_\odot$, we obtain $\theta_\mathrm{E} = 1.8$ mas, $u = 0.01/1.8 = 0.00552$, and thus, $\underline{\mu = 181}$. With $R_\mathrm{E} = \theta_\mathrm{E}D_\mathrm{L} = 5.4\times10^{13}$ cm and an assumed velocity $v \approx 200$ km/s, the transit duration is $\underline{t = 31.2\ \mathrm{d}}$.

b) $\mathscr{M} = 1\ \mathscr{M}_\mathrm{Jup}$ leads to: $\theta_\mathrm{E} = 0.0556$ mas, $u = 0.01/0.0556 = 0.18$, $\underline{\mu = 5.63}$, $R_\mathrm{E} = 1.66\times10^{12}$ cm, $\underline{t = 23.1\ \mathrm{h}}$.

c) $\mathscr{M} = 1\ \mathscr{M}_\oplus$ leads to: $\theta_\mathrm{E} = 1.5\times10^{-11}$ rad $= 0.00312$ mas, $u = 0.01/0.00312 = 3.2$, $\underline{\mu = 1.013}$, $R_\mathrm{E} = 9.24\times10^{10}$ cm, $\underline{t = 1.3\ \mathrm{h}}$.

Problem 8.2 *(2 points)*

Imagine you measure the arrival times of pulses from a pulsar (with $\mathscr{M}_* = 1.4\ \mathscr{M}_\odot$) and you note that the times deviate periodically (with a period $P = 1$ yr) by up to ±1 ms from those expected for constant intervals. What is the minimum mass of a possible companion that could cause this deviation. *Hint: assume a circular orbit.*

Neglecting relativistic effects, the true times at which the pulsar emits its pulses are given by

$$t_n = t_0 + nP. \tag{3}$$

In contrast, the times at which the pulses arrive at the barycenter of the solar system are

$$t'_n = t_n + \frac{r(t_n)}{c}, \tag{4}$$

where $r(t_n)$ is the distance at the time of pulse emission and c the speed of light. The radial velocity of the pulsar is composed of the (near-constant) system velocity and the variation due to orbital motion around the barycenter:

$$\dot{r} = v_\mathrm{r} = v_\mathrm{sys} + \Delta v_\mathrm{r}(t). \tag{5}$$

The resulting distance is

$$r = r_0 + \int_{t_0}^{t} v_\mathrm{r}\mathrm{d}t \;=\; r_0 + (t - t_0)v_\mathrm{sys} + \Delta r(t), \tag{6}$$

where Δr is the pulsar's distance from the common barycenter with its companion. Hence we find

$$\begin{aligned} t'_n &= t_n + \frac{r_0 + (t_n - t_0)v_\mathrm{sys} + \Delta r(t_n)}{c} \\ &\overset{\text{eq. (3)}}{=} t_0 + nP + \frac{r_0 + nPv_\mathrm{sys} + \Delta r(t_n)}{c} \\ &= t'_0 + n\underbrace{P\left(1 + \frac{v_\mathrm{sys}}{c}\right)}_{\equiv P'} + \underbrace{\frac{\Delta r(t_n) - \Delta r(t_0)}{c}}_{\equiv \Delta t'_n}. \end{aligned} \tag{7}$$

If no companion were present, we would expect the pulses to arrive at times $t'_0 + nP'$, i. e. with constant intervals P'. However the radial displacement (Δr) causes a variation in light travel times, and hence, the pulse timings from those expected, $\Delta t'_\mathrm{n}$. The maximum (semi-)amplitudes of radial displacement and timing variations are related to the semi-major axis of the pulsar's orbit, a_pulsar, via:

$$\Delta t'_\mathrm{max} = \frac{\Delta r_\mathrm{max}}{c} = \frac{a_\mathrm{pulsar} \sin i}{c}, \tag{8}$$

where i is the inclination of the pulsar–companion orbit relative to the plane of the sky. Using Kepler's third law,

$$P_\mathrm{orb} = 2\pi\sqrt{\frac{a^3}{G\mathcal{M}}} \tag{9}$$

with $\mathcal{M} = \mathcal{M}_\mathrm{pulsar} + \mathcal{M}_\mathrm{comp}$, and the definition of the barycenter,

$$a = a_\mathrm{pulsar} + a_\mathrm{comp} = a_\mathrm{pulsar}\left(1 + \frac{\mathcal{M}_\mathrm{pulsar}}{\mathcal{M}_\mathrm{comp}}\right), \tag{10}$$

we obtain

$$P_\mathrm{orb} = 2\pi\sqrt{\frac{a^3_\mathrm{pulsar}\left(1 + \frac{\mathcal{M}_\mathrm{pulsar}}{\mathcal{M}_\mathrm{comp}}\right)^3}{G(\mathcal{M}_\mathrm{pulsar} + \mathcal{M}_\mathrm{comp})}} \approx 2\pi\sqrt{\frac{a^3_\mathrm{pulsar}\mathcal{M}^2_\mathrm{pulsar}}{G\mathcal{M}^3_\mathrm{comp}}} \overset{\text{eq. (8)}}{=} 2\pi\sqrt{\frac{(c\,\Delta t'_\mathrm{max})^3\,\mathcal{M}^2_\mathrm{pulsar}}{G\left(\mathcal{M}_\mathrm{comp}\sin i\right)^3}}, \tag{11}$$

and after solving for the minimum mass,

$$\mathcal{M}_\mathrm{comp}\sin i = c\,\Delta t_\mathrm{max}\sqrt[3]{\frac{1}{G}\left(\frac{2\pi\mathcal{M}_\mathrm{pulsar}}{P_\mathrm{orb}}\right)^2}. \tag{12}$$

Assuming $\mathcal{M}_\mathrm{pulsar} = 1.4\ \mathcal{M}_\odot$, $P_\mathrm{orb} = 1$ yr, and $\Delta t'_\mathrm{max} = 1$ ms, we find

$$\mathcal{M}_\mathrm{comp}\sin i = 5\times 10^{24}\ \mathrm{kg} \approx 0.8\,\mathcal{M}_\mathrm{Earth}, \tag{13}$$

i. e. we may have detected an Earth-mass planet (if the inclination is not too far away from 90°).

Extra info: the difference between emitted and observed pulse periods P and P', respectively, is due to the simple, "acoustic" doppler effect.

Problem 8.3 *(3 points)*

The 3D equation of mass growth rate is:

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}t} \approx \rho\sigma v_{\mathrm{rel}} \tag{14}$$

In 2D, the following changes are needed. Firstly, ρ is replaced by surface density, Σ. Secondly, the cross section for collision σ without gravitational enhancement is $2s$ instead of πs^2. With gravitational enhancement, it is therefore

$$\sigma = 2s\left(1+\frac{v_{\mathrm{esc}}^2}{v_{\mathrm{rel}}^2}\right)^{1/2} \tag{15}$$

instead of

$$\sigma = \pi s^2\left(1+\frac{v_{\mathrm{esc}}^2}{v_{\mathrm{rel}}^2}\right). \tag{16}$$

Collecting all results together and using $v_{\mathrm{rel}} \ll v_{\mathrm{esc}}$ as in the 3D case, we obtain the 2D equation of mass growth rate:

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}t} \approx \Sigma \cdot 2s \cdot v_{\mathrm{esc}} \tag{17}$$

or

$$\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}t} \propto \Sigma\mathscr{M}^{2/3} \tag{18}$$

where the power of the mass in the right-hand side is less than unity, so it is not a runaway growth. The exponent (2/3) is the same as for the oligarchic growth.

Problem 8.4 *(2 points)*

The mass of finished oligarchs is given by

$$\mathscr{M}_{\mathrm{iso}} = \frac{(2\pi b\Sigma)^{3/2} r^3}{(3\mathscr{M}_*)^{1/2}} \tag{19}$$

Assume $b = 10$ and $\Sigma = 10$ g cm^{-2} at 1 au. Then

$$\begin{aligned}\mathscr{M}_{\mathrm{iso}} &= \frac{(2\cdot 3\cdot 10\cdot 10)^{3/2}(1.5\cdot 10^{13})^3}{(3\cdot 2\cdot 10^{33})^{1/2}} \approx \frac{600^{3/2}\cdot 3\cdot 10^{39}}{(60\cdot 10^{32})^{1/2}} \approx \frac{600\cdot 25\cdot 3\cdot 10^{39}}{8\cdot 10^{16}} \approx \frac{600\cdot 10\cdot 10^{39}}{10^{16}} \\ &\approx 6\cdot 10^{26}\ \mathrm{g} \approx 0.1\mathscr{M}_\oplus\end{aligned}$$

To get the result at 5 au, we have to multiply this by $(3/10)^{3/2}$ (to account for difference in Σ) and by $(5/1)^3$ (to account for difference in r). This gives:

$$\mathscr{M}_{\mathrm{iso}} \approx 0.1\mathscr{M}_\oplus \cdot (3/10)^{3/2}\cdot 5^3 \approx 0.1\mathscr{M}_\oplus \cdot 20 \approx 2\mathscr{M}_\oplus. \tag{20}$$

The orbital separation of isolated oligarchs is given by

$$\Delta r = br_{\mathrm{H}} = br\left(\frac{\mathscr{M}_{\mathrm{iso}}}{3M_*}\right)^{1/3} \tag{21}$$

Numerically, at 1au

$$\Delta r \approx 10 \cdot 1.5 \cdot 10^{13} \left(\frac{6 \cdot 10^{26}}{3 \cdot 2 \cdot 10^{33}} \right)^{1/3} \approx 1.5 \cdot 10^{14} \left(\frac{1}{10 \cdot 10^{6}} \right)^{1/3} \approx 10^{12} \text{ cm} \approx 0.07\text{au} \tag{22}$$

Again, to get the result at 5 au, we have to multiply this by 5 (to account for the difference in r) and with $20^{1/3}$ (to account for difference in $\mathscr{M}_{\text{iso}}$). This gives:

$$\Delta r \approx 0.07 \text{ au} \cdot 5 \cdot 2.7 \approx 1 \text{ au} \tag{23}$$