Physics of Planetary Systems — Exercises
Suggested Solutions to Set 9

Problem 9.1 *(2 points)*

To address the question of how planetary radii and temperatures are related, we need the temperatures. Measuring the temperatures is very difficult or completely unfeasible is most cases, so that we have to rely on estimates based on the typical irradiation from the star, depends on the stellar luminosity, L_*, and the star-planet distance, r. Equating absorbed stellar radiation (over the cross section of the planet) and emitted thermal radiation (over its whole surface), we find

$$\begin{aligned} L_{\rm in} &\stackrel{!}{=} L_{\rm out} \\ \frac{L_*}{4\pi r^2}\pi R_{\rm p}^2(1-A) &= 4\pi R_{\rm p}^2\sigma T_{\rm p}^4 \\ \frac{4\pi\sigma R_*^2 T_*^4}{4\pi r^2}\pi R_{\rm p}^2(1-A) &= 4\pi R_{\rm p}^2\sigma T_{\rm p}^4 \\ \frac{R_*^2 T_*^4}{r^2}(1-A) &= 4T_{\rm p}^4 \\ \sqrt{\frac{R_*}{2r}}\sqrt[4]{1-A}\ T_* &= T_{\rm p}, \end{aligned} \tag{1}$$

where R_* and T_* are the stellar radius and temperature, respectively, and A the planet's Bond albedo. In solar units, we have

$$T_{\rm p} = 279\ {\rm K} \times T_*[T_\odot]\sqrt{\frac{R_*[R_\odot]}{2r[{\rm au}]}}, \tag{2}$$

where we assumed $\sqrt[4]{1-A} \approx 1$. After extracting the tabulated values for $M_{\rm p}$, $R_{\rm p}$, R_*, $T_*(=T_{\rm eff})$, and r(assumed $\approx a$) from `https://exoplanet.eu/`, we can filter for $M_{\rm p} > 0.3M_{\rm Jup}$, calculate $T_{\rm p}$ and plot $T_{\rm p}$ vs $R_{\rm p}$.

The resulting distribution is shown in Fig. 1. For low temperatures ($T_{\rm p} \lesssim 1300$ K), all planets have their radii close to the Jupiter radius, which is expected from electron degeneracy pressure in equilibrium with gravity. For higher temperatures ($T_{\rm p} \gtrsim 1300$ K) the extra heat from the stars puffs the outer planet envelopes up, leading to drasticly increased radii.

Extra info: we could have made our lives a bit easier by plotting $T_{\rm p}$ vs $R_{\rm p}$ directly on the `exoplanet` website because the database already provides the calculcated temperatures.

Problem 9.2 *(2 points)*

Based on the measured radii and masses of stars and substellar companions shown in Fig. 2 (and in the lecture notes), we can deduce the following relations:

(a) $3R_{\rm Jup} \implies (150\ldots400)\mathscr{M}_{\rm Jup} = (0.15\ldots0.4)\mathscr{M}_\odot \implies$ stars,
(b) $1R_{\rm Jup} \implies (1\ldots100)\mathscr{M}_{\rm Jup} \implies$ gas giants, brown dwarfs, or low-mass stars,
(c) $0.3R_{\rm Jup} \implies (0.01\ldots0.05)\mathscr{M}_{\rm Jup} = (3\ldots15)\mathscr{M}_\oplus \implies$ super-earths and neptunes, and
(d) $0.1R_{\rm Jup} \implies \lesssim 5\mathscr{M}_\oplus \implies$ sub-earths to super-earths.

Bonus problem 9.3 *(2 extra points)*

Following up on a previous problem, we can start with the orbital separation of isolated oligarchs:

$$\Delta r = br_{\rm H} = br\left(\frac{\mathscr{M}_{\rm iso}}{3\mathscr{M}_*}\right)^{1/3}, \tag{3}$$

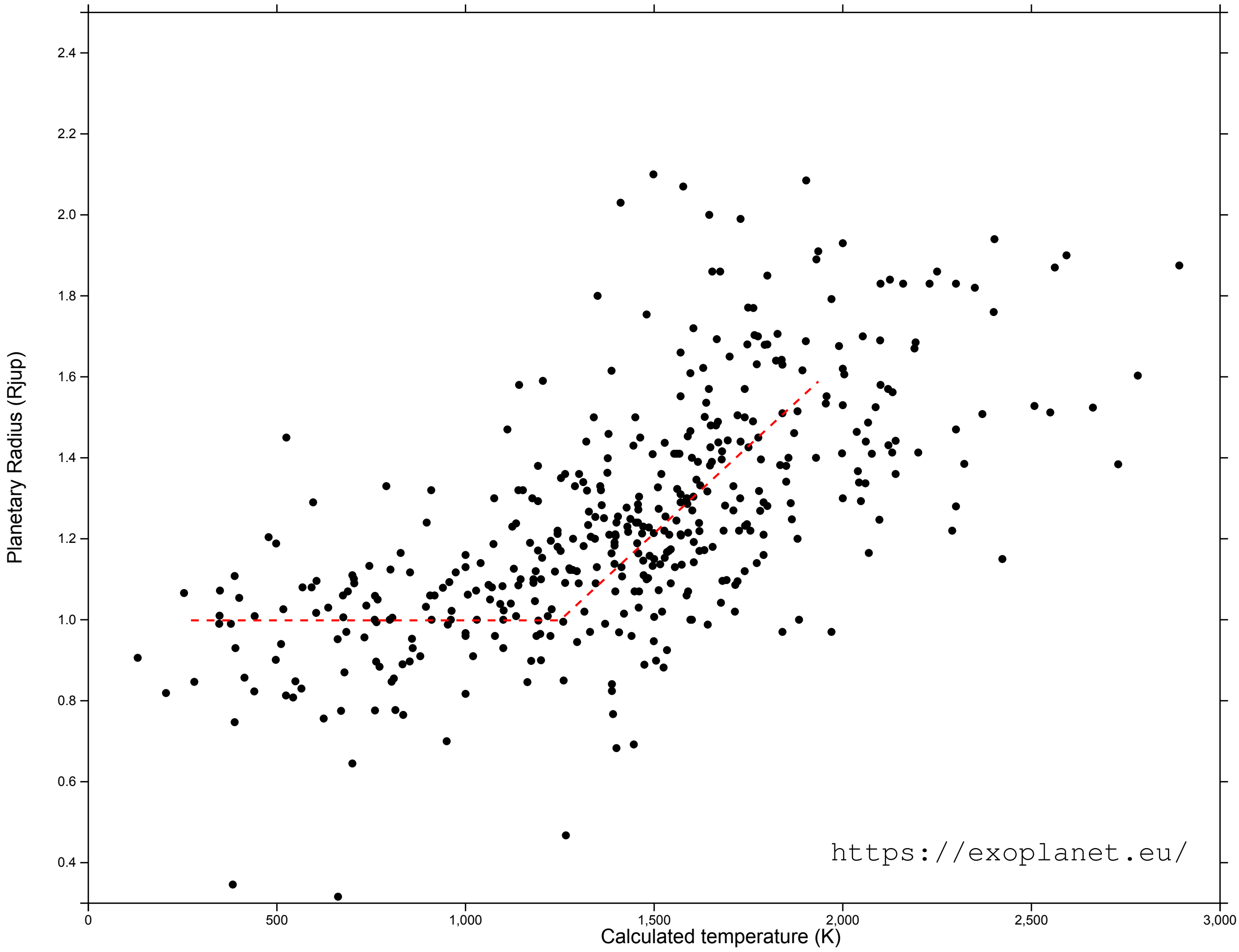

Figure 1: Radii vs calculated temperatures of confirmed planets with masses $M_\mathrm{p} > 0.3 M_\mathrm{Jup}$. Dashed lines indicate the two rough temperature regimes.

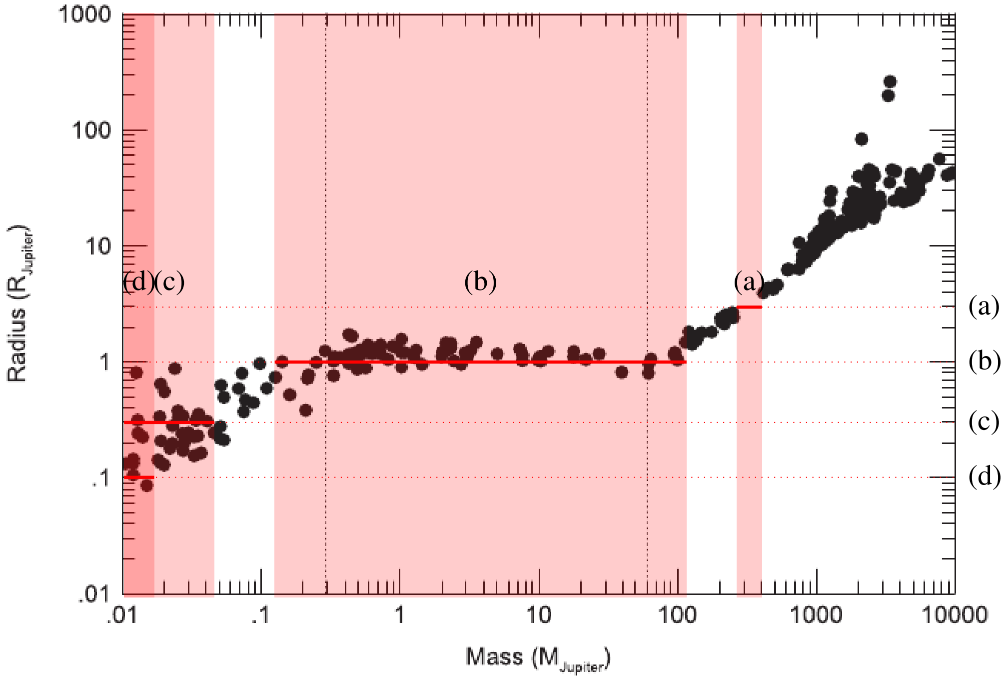

Figure 2: Measured radii and masses of stars and substellar companions as presented by Hatzes & Rauer (2015; see also the lecture notes).

where

$$\mathscr{M}_{\text{iso}} = \frac{(2\pi b\Sigma)^{3/2} r^3}{(3\mathscr{M}_*)^{1/2}} \tag{4}$$

is the isolation mass, such that the separation becomes

$$\Delta r = br^2 \left(\frac{2\pi b\Sigma}{3\mathscr{M}_*}\right)^{1/2} = br^2 \left(\frac{2\pi b\Sigma_0}{3\mathscr{M}_*}\right)^{1/2} \left(\frac{r}{r_0}\right)^{-3/4}, \tag{5}$$

where we assumed a power-law density distribution,

$$\Sigma = \Sigma_0 (r/r_0)^{-3/2}, \tag{6}$$

with reference density Σ_0 at reference distance r_0.

Thus, if we move outward in the disk, we pass one oligarch per Δr, i. e. the total number N increased by one for each Δr, or:

$$\frac{\mathrm{d}N}{\mathrm{d}r} \approx \frac{\Delta N}{\Delta r}. \tag{7}$$

For the total number in the range from r_{in} to r_{out}, we obtain

$$\begin{aligned} N &= \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{\mathrm{d}N}{\mathrm{d}r}\mathrm{d}r = \int_{r_{\text{in}}}^{r_{\text{out}}} \frac{1}{br^2}\left(\frac{r}{r_0}\right)^{3/4} \sqrt{\frac{3\mathscr{M}_*}{2\pi b\Sigma_0}}\,\mathrm{d}r = \frac{1}{br_0^{3/4}} \sqrt{\frac{3\mathscr{M}_*}{2\pi b\Sigma_0}} \int_{r_{\text{in}}}^{r_{\text{out}}} r^{-5/4}\,\mathrm{d}r \\ &= \frac{-4}{br_0^{3/4}} \sqrt{\frac{3\mathscr{M}_*}{2\pi b\Sigma_0}} \left[\frac{1}{r_{\text{out}}^{1/4}} - \frac{1}{r_{\text{in}}^{1/4}}\right] = 4r_0 \underbrace{\frac{1}{br_0^2}\sqrt{\frac{3\mathscr{M}_*}{2\pi b\Sigma_0}}}_{1/\Delta r(r=r_0)} \left[\left(\frac{r_0}{r_{\text{in}}}\right)^{1/4} - \left(\frac{r_0}{r_{\text{out}}}\right)^{1/4}\right]. \end{aligned} \tag{8}$$

After setting the arbitrary reference radius $r_0 = r_{\text{in}}$, we find

$$N = \frac{4r_{\text{in}}}{\Delta r(r_{\text{in}})} \underbrace{\left[1 - \left(\frac{r_{\text{in}}}{r_{\text{out}}}\right)^{1/4}\right]}_{\approx 0.6} \sim \frac{4r_{\text{in}}}{\Delta r(r_{\text{in}})}. \tag{9}$$

where $r_{\text{in}} = 1$ au and $r_{\text{out}} = 30$ au.

With our estimated $\Delta r = 0.07$ au at $r_{\text{in}} = 1$ au (see previous problem set), the final result is

$$N \approx 30. \tag{10}$$

Extra info: note that the factor 4 ultimately comes from the exponent in eq. (6). If we assumed the same density at the inner edge but a shallower slope, say -1 instead of $-3/2$, we would end up with fewer (but more massive) oligarchs – and vice versa. For a more general density distribution, $\Sigma = \Sigma_0 (r/r_0)^{-x}$, with a slope $x \neq 2$, we would obtain

$$N = \frac{1}{1-\frac{x}{2}} \frac{r_{\text{in}}}{\Delta r(r_{\text{in}})} \left[1 - \left(\frac{r_{\text{in}}}{r_{\text{out}}}\right)^{1-\frac{x}{2}}\right]. \tag{11}$$

Problem 9.4 *(2 points)*

We can solve the known equation for the (blackbody) equilibrium temperature,

$$T = T_* \sqrt{\frac{R_*}{d}} \sqrt[4]{\frac{1-A}{4}}, \tag{12}$$

for the distance d to obtain

$$d = R_* \left(\frac{T}{T_*}\right)^{-2} \sqrt{\frac{1-A}{4}}. \tag{13}$$

Assuming $T_* = T_\oplus = 5800$ K, $R_* = R_\oplus \approx 7 \times 10^8$ m, $A \approx 0$, along with $T_{\mathrm{H_2O}} = 170$ K and $T_{\mathrm{CO}} = 20$ K, the resulting distances to the icelines are

$$d_{\mathrm{H_2O}} \approx 2300\, R_\oplus \approx 2.7 \text{ au} \qquad \text{and} \qquad d_{\mathrm{CO}} \approx 42\,000\, R_\oplus \approx 200 \text{ au}. \tag{14}$$

These are only rough estimates because both temperatures, for thermal equilibrium and sublimation, depend on other parameters (such as the albedo and the actual gas pressure/density) and can therefore vary over time or from system to system. For example, a grain of pure water ice will have a much higher albedo (and lower equ. temperature) than an ice grain that is contaminated by “dirt”. Regardless of these details, the H_2O ice line can be important for planetesimal formation in the terrestrial region, while the CO ice line could be more relevant for the “outer edges” (such as the Kuiper belt). Whether ice lines have a dominant effect on the final structure of systems is still an open question.